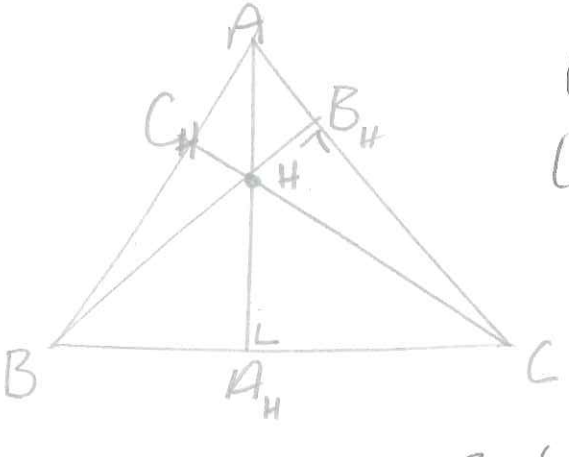
Maths Enrichment, UCD 25th February '16

Geometry

The material covered today relies on a knowledge and understanding of the definitions and theorems (and their corollaries) in section 6.11 (Circles), on pages 80-86 of the Leaving Cert. Mathematics Syllabus of An Roinn Oideachais agus Scileanna. This is available online — it was also handed out during the lecture.

Example: Let ABC be an acute-angled triangle and let $[AA_H]$, $[BB_H]$ and $[CH_H]$ be its altitudes. We denote the orthocentre by H.



Prove that

(1) $C_H B_H C B$ is a cyclic quadrilateral

(2) $C_H H A_H B$ " " "

Proof of (1):

$$|\angle BC_HC| = |\angle BB_HC| = 90^\circ$$

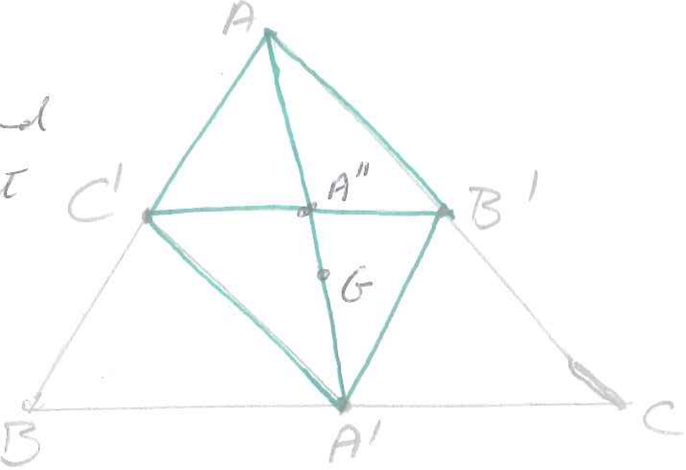
so (1) follows by Cor 2 (converse) (page 81) of Theorem 19 in above.

Proof of (2) $|\angle BC_HH| = 90^\circ = |\angle HA_HB|$ so the result follows by the converse of Cor. 5 (page 82) of Theorem 19 above.

We denote the medians of $\triangle ABC$ by $[AA']$, $[BB']$ and $[CC']$ and its centroid by G. The $\triangle A'B'C'$ is called the medial $\triangle$ of $\triangle ABC$.

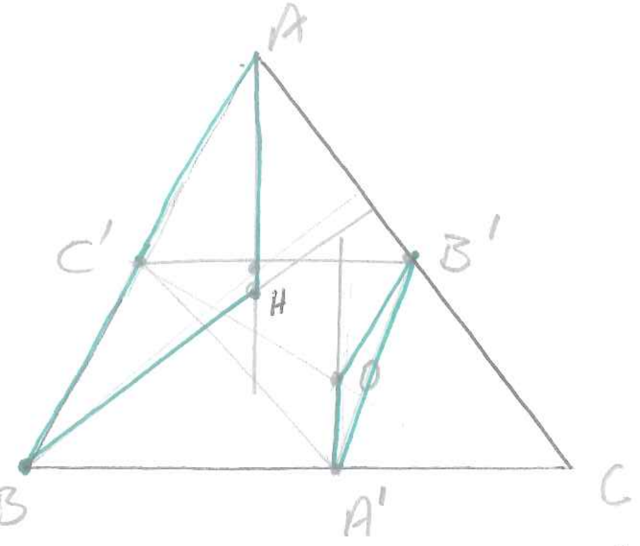
Lemma: A triangle and its medial $\triangle$ have the same centroid.

Proof: We show that G is the centroid of $\triangle A'B'C'$.



Since C' is the midpt. of $[AB]$ and B' " " " " $[AC]$, it follows that $C'B' \parallel BC$ and $|C'B'| = \frac{1}{2}|BC|$.

Similarly, $C'A' \parallel AC$ and $A'B' \parallel BA$. Thus, $AB'A'C'$ is a parallelogram, so the diagonal AA' bisects $[C'B']$, at A'' say. Hence, $[A'A'']$ is a median of $\triangle A'B'C'$ and we note that this line segment lies on the median ~~of~~ $[AA']$ of $\triangle ABC$ and contains the point G. By the same argument, the other two medians of $\triangle A'B'C'$ lie along the medians of $\triangle ABC$. We conclude that the centroid of $\triangle A'B'C'$ is the point G.



We denote the circumcentre of ΔABC by O.

Note that the perpendicular bisectors of the sides of ΔABC coincide with the altitudes of $\Delta A'B'C'$.

We conclude that the circumcentre of ΔABC (i.e. O) is the orthocentre of $\Delta A'B'C'$.

Lemma: $|AH| = 2|A'O|$.

Proof: In the Δs AHB and $A'OB'$,

$AH \parallel A'O$ (both $\perp$ BC),

$BH \parallel B'O$ (" $\perp$ AC),

and $AB \parallel B'A'$ so these Δs are similar and so their sides are in proportion.

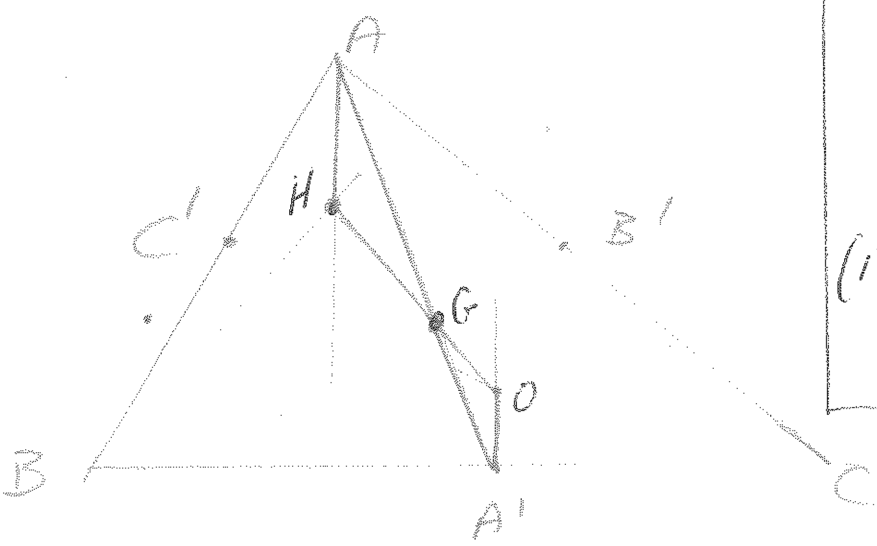
Now $|AB| = 2|A'B'|$, so it follows that

$|AH| = 2|A'O|$

We will use this result in the next important theorem.

Theorem: The orthocentre, centroid and circumcentre of a triangle are collinear.

This line is called the Euler line.
Leonhard Euler (1707–1783) was born in Basel, Switzerland.



We want to show that H, G and O are collinear. We will consider the $\triangle$s AHG and $A'OG$. [We know that G is on the line AA'. If we can show that these $\triangle$s are similar then it will follow that $\angle AGH = \angle A'GO$ and so G must lie on the line HO also.]

Now, $AH \parallel OA'$ (both $\perp BC$) and
$|AH| = 2|A'O|$, from the last lemma.

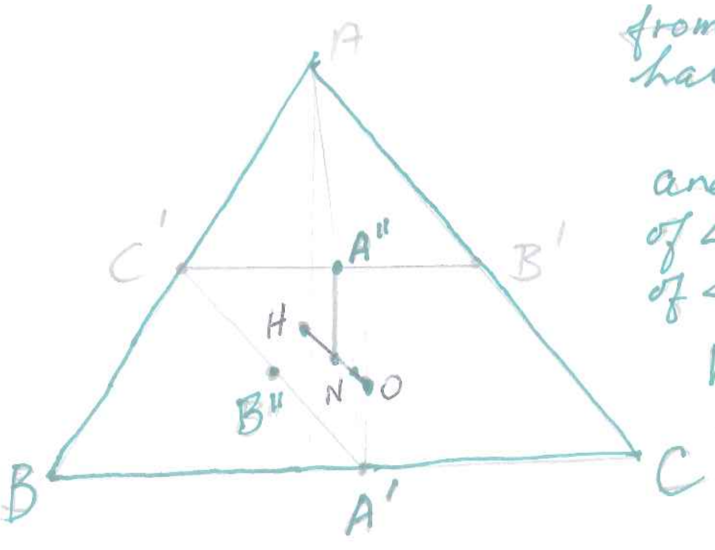
Also AG and GA' are collinear with
$|AG| = 2|GA'|$ (the centroid divides a median in the ratio $2:1$).

We conclude that $\triangle$s AHG and $A'OG$ are similar and the result follows.

Note: we can also conclude that
$$\boxed{|HG| = 2|GO|.}$$

Theorem:

The circumcentre of the medial $\triangle A'B'C'$ lies on the Euler line.



This follows immediately from: (1) $\triangle ABC$ and $\triangle A'B'C'$ have the same centroid G (page 2)

and: (2) the circumcentre of $\triangle ABC$ is the orthocentre of $\triangle A'B'C'$ (page 3).

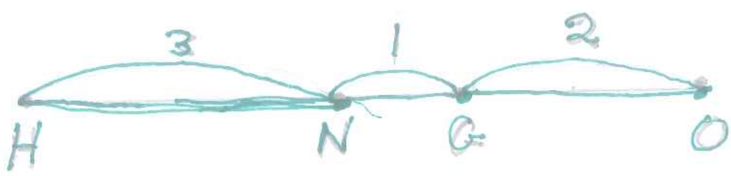
We give another proof below.

Proof: We let A'' denote the midpoint of $[C'B']$ and draw $A''N \perp B'C'$ meeting OH at N. (We note that the circumcentre of $A'B'C'$ lies on the line $A''N$.) Now, $|AA''| = |A''A'|$, so the three perpendiculars to BC through the points A, A'' and A', that is, AH, $A''N$ and $A'O$, are evenly spaced. Thus, $|HN| = |NO|$.

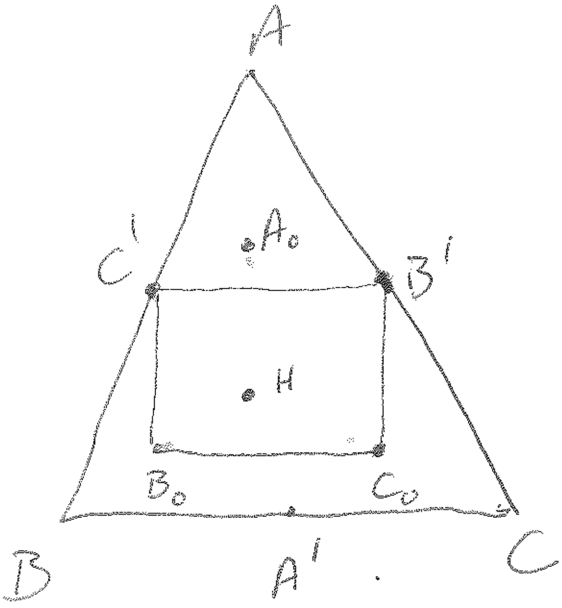
Similary, perpendiculars to AC through B, B'' and B' are evenly spaced, so the perpendicular bisector of $[C'A']$ meets $[HO]$ at N, its midpoint. Hence, N is on the perpendicular bisectors of all three sides of $\triangle A'B'C'$; that is, the circumcentre of $\triangle A'B'C'$ is on the Euler line.

Note:

we now have



The Euler Line



Let A_0 be the midpoint of $[AH]$,
B_0 " " " " $[BH]$,
C_0 " " " " $[CH]$.

Lemma: The following three quadrilaterals are rectangles:
$B_0C_0B'C'$, $B_0A_0B'A'$ and $A_0C_0A'C'$.

Proof: Consider the $\triangle ABH$.

$$\left.\begin{array}{l}\text{The midpoint of } [AB] \text{ is } C' \\ \text{" " " } [HB] \text{ is } B_0\end{array}\right\} \Rightarrow C'B_0 \parallel AH.$$

Similarly, if we consider the $\triangle ACH$, we see that $B'C_0 \parallel AH$. So, $C'B_0 \parallel B'C_0$.
The same argument applied to $\triangle ABC$ and $\triangle HBC$ shows that

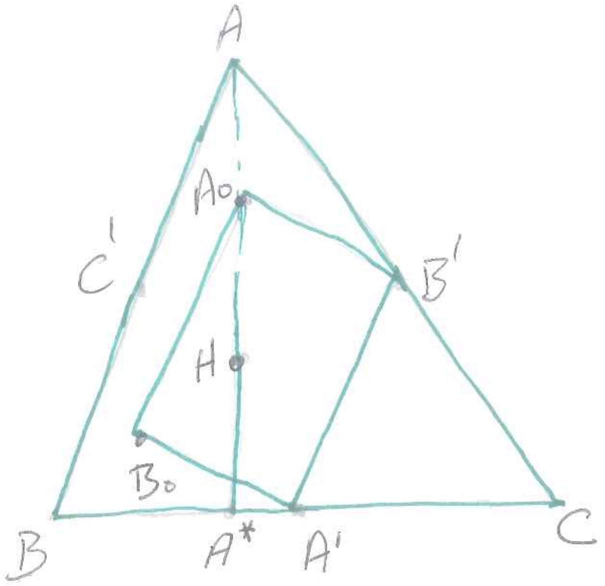
$$C'B' \parallel BC \text{ and } B_0C_0 \parallel BC.$$

Thus $B_0C_0B'C'$ is a parallelogram.
Since $AH \perp BC$ it follows that
$B_0C_0B'C'$ is a rectangle.

Having proved this lemma, it is easy to show that:

Theorem: The points A_0, B_0 and C_0 lie on the circumcircle of the medial $\triangle A'B'C'$!

Proof: A rectangle is a cyclic quadrilateral by the converse of Cor. 5 (on page 82) of Theorem 19. Its diagonals are the diameters of the circle containing its vertices (Cor. 3, p 8).
Since $B_0C_0B'C'$ is a rectangle $[C_0C']$ is a diameter of the circle containing the points B_0, C_0, B' and C'.
Also, ~~A~~ $[C_0C']$ is a diameter of the circle containing the points A_0, C_0, A', C'.
Now, $[C_0C']$ is a diameter of one unique circle, so A', B', C', A_0, B_0 and C_0 lie on the same circle — the circumcircle of the medial $\triangle$.



We denote the altitudes of the $\triangle ABC$ by $[AA^*]$, $[BB^*]$ and $[CC^*]$ and refer to the points A^*, B^* and C^* as the feet of the altitudes

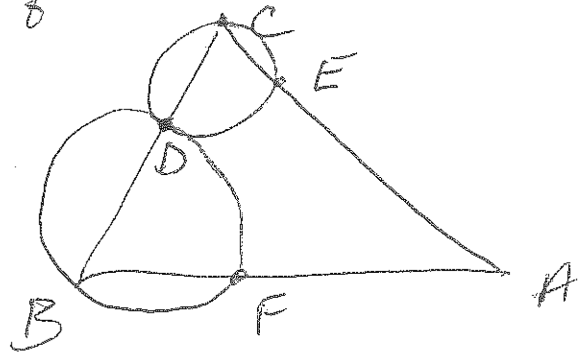
We make use of the last theorem to show that:

Theorem. The feet of the altitudes of a triangle lie on the circumcircle of its medial triangle.

Proof: We will show that A^* lies on this circle. We know from the previous page that $A_0B_0A'B'$ is a rectangle and that $[A_0A']$ is therefore a diameter of the circumcircle of $\triangle A'B'C'$.
The altitude AA^* is perpendicular to BC so $\angle A_0A^*A'$ is a right-angle. By the converse of Cor 3 (page 82), A^* must lie on the circumcircle of $\triangle A'B'C'$ (since $[A_0A']$ is a diameter.
Similarly, we can show that B^* and C^* lie on this circle.

The nine-point circle is also called Feuerbach's circle. F born in Jena (1800 – 1834)

Problem 1 (a) The $\triangle ABC$ has three acute angles. Let D be any point on the side $[CB]$. The circle Γ_C thro' the points C and D and the ⊙ Γ_B " " " B " D touch at the point D. The ⊙ Γ_C intersects $[CA]$ at the point E and Γ_B intersects $[BA]$ at F. Show that D is on the circumcircle of $\triangle EAF$.



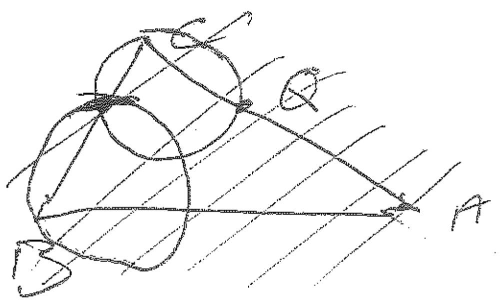
Solution:

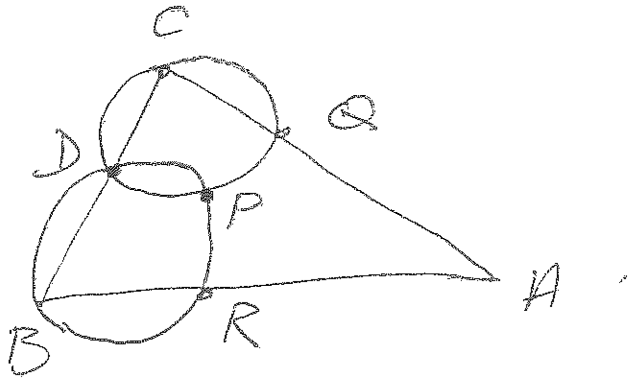
Since Γ_C and Γ_B touch, $[CD]$ is a diameter of Γ_C

$\Rightarrow |C\hat{E}D| = 90°$, and $[BD]$ " " " " Γ_B

$\Rightarrow |D\hat{F}B| = 90°$. So $|\angle DEA| = 90° = |\angle DFA|$ and these are opposite angles of the quadrilateral $EDFA$. Hence it is a cyclic quadrilateral.

Problem (b) Again, $\triangle ABC$ has three acute angles, D is any point on $[CB]$, but this time, the $\mathcal{C}_C$ through C and D, and the circle $\mathcal{C}_B$ thro' B and D intersect at two points D and P where P is inside the $\triangle ABC$. The $\odot$ $\mathcal{C}_C$ intersects $[CA]$ at Q and the circle $\mathcal{C}_B$ " $[BA]$ at R. Show that P is on the circumcircle of the $\triangle RAQ$.





Solution: The quadrilateral $DPRB$ is cyclic $\Rightarrow$ $|D\hat{P}R| = 180^\circ - |D\hat{B}R|$. Since $CDPQ$ is cyclic also, $|D\hat{P}Q| = 180^\circ - |D\hat{C}Q|$.

$$
\begin{aligned}
\text{Now } |Q\hat{P}R| &= 360^\circ - (|D\hat{P}R| + |D\hat{P}Q|) \\
&= |D\hat{B}R| + |D\hat{C}Q| \\
&= 180^\circ - |C\hat{A}B| \\
&= 180^\circ - |Q\hat{A}R|.
\end{aligned}
$$

So $QPRA$ is cyclic.

Popular maths books

The books listed here can be enjoyed by people who have an interest in maths without having studied maths at third level.

- *The Simpsons and Their Mathematical Secrets* (2013), by Simon Singh,
- *Alex's Adventures in Numberland* (2010), by Alex Bellos,
- *Fermat's Last Theorem* (1997), by Simon Singh,
- *The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography* (2000), by Simon Singh,
- *The Music of the Primes* (2003), by Marcus du Sautoy.

Mary Hanley